Asymmetric unimodal maps: Some results from q-generalized bit cumulants

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In this study, using q-generalized bit cumulants (q is the nonextensivity parameter of the recently introduced Tsallis statistics), we investigate the asymmetric unimodal maps $x_{t+1}=1-a|x_t|^{z_i}$ ($i=1,2$ correspond to $x_t>0$ and $x_t<0$, respectively; $z_i>1$, $0<a\leq 2$, $t=0,1,2,\ldots$). The study of the q-generalized second cumulant $C_2^{(q)}$ of these maps allows us to determine the dependence of the nonextensivity parameter q on the inflection parameter pairs (z_1,z_2). The slope of the $C_2^{(q)}$ versus $C_2^{(1)}$ plot (where $C_2^{(1)}$ is the standard second cumulant) provides the necessary tool to accomplish this task. The slope behaves exactly the same as the proper q values (say q^*) that were obtained for logisticlike maps ($z_1=z_2=z$) by Costa *et al.* [Phys. Rev. E **56**, 245 (1997)]. It appears that as $z_2-z_1\rightarrow\pm\infty$ this slope approaches unity. This behavior is very similar to the behavior of q^* as a function of the inflection parameter for some z-dependent maps; namely, as $z\rightarrow\infty$, q^* approaches 1.

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In recent years, the sensitivity to initial conditions of nonlinear dynamical systems has been studied with increasing interest. As examples, dissipative systems of low-dimensional maps [1–4], self-organized criticality [5], symbolic sequences [6] and conservative systems of long-ranged many-body Hamiltonians [7,8], and low-dimensional conservative maps [9] can be enumerated. We focus here on one-dimensional dissipative maps. As is well known, to investigate the sensitivity to initial conditions of any kind of one-dimensional map at the onset of chaos, it is possible to introduce

$$\xi(t)=\lim_{\Delta x(0)\rightarrow 0}\frac{\Delta x(t)}{\Delta x(0)}, \tag{1}$$

where $\Delta x(0)$ and $\Delta x(t)$ are discrepancies of the initial conditions at times 0 and t. (Here it is worth mentioning that in general x_i is a function of time, i, and initial position x_0, but since we shall be interested in dissipative maps in this study, it is essentially included in the attractor. Moreover, in our calculations, we choose $x_0=0$ which belongs to the attractor.) In general, $\xi(t)$ satisfies the differential equation $d\xi/dt=\lambda_1\xi$; thus

$$\xi(t)=e^{\lambda_1 t}, \tag{2}$$

where λ_1 is the Lyapunov exponent. However, for the marginal case $\lambda_1=0$, it satisfies $d\xi/dt=\lambda_q\xi^q$; thus

$$\xi(t)=[1+(1-q)\lambda_q t]^{1/(1-q)}(q\in\mathcal{R}), \tag{3}$$

which recovers the standard case [namely, Eq. (2)] for $q\rightarrow 1$. (Although in principle q is allowed to be any real number, for the one-dimensional maps studied so far, we see that q is below unity.) Here, q is the nonextensivity parameter of the recently introduced Tsallis statistics [10] and for $q\neq 1$ it is evident that Eq. (3) yields a *power-law* sensitivity to initial conditions. This kind of power-law behavior was observed previously in [11]. At the onset of chaos, $\xi(t)$ presents strong fluctuations with time, reflecting the fractal-like structure of the critical attractor, and Eq. (3) delimits the power-law growth of the upper bounds of $\xi(t)$. [Of course, if we had chosen x_0 different from 0, but still on the attractor, we would have seen a different $x_i(t)$, but it still fluctuates similarly.] These upper bounds ($\xi\propto t^{1/(1-q)}$) allow us to calculate a single value of q (say q^*) for the map under consideration. This method of finding numerical values of q^* has been successfully used for the logistic map [1], a family of logisticlike maps [2], the circle map [3], and a family of circularlike maps [4]. Besides this method, another one has been developed by Lyra and Tsallis [3] by looking at the geometrical aspects of dynamical attractors at the threshold of chaos. Using the multifractal singularity spectrum $f(\alpha)$ [12], they proposed the scaling relation

$$\frac{1}{1-q^*}=\frac{1}{\alpha_{min}}-\frac{1}{\alpha_{max}}, \tag{4}$$

where α_{max} (α_{min}) is the most rarefied (concentrated) region of the multifractal singularity spectrum of the attractor. This relation presents the second method of calculating the q^* values once the scaling properties of the dynamical attractor are known. Here we should mention that the present index q is not the same as the parameter q of Ref. [12], which is a free parameter used to probe the multifractals using generalized Renyi entropies. It is also worth noting that the nonextensive Tsallis entropy is completely different from the Renyi entropy in the sense that the former has definite concavity for all values of q, whereas the latter does not. For example, it was very recently shown in [13,14] that the time evolution of the Tsallis entropy can be linear for one and only one value of the q index for a given value of the inflection number z, which coincides with the q^* values calculated from the above-mentioned two methods, whereas this is in general false for Renyi entropies.

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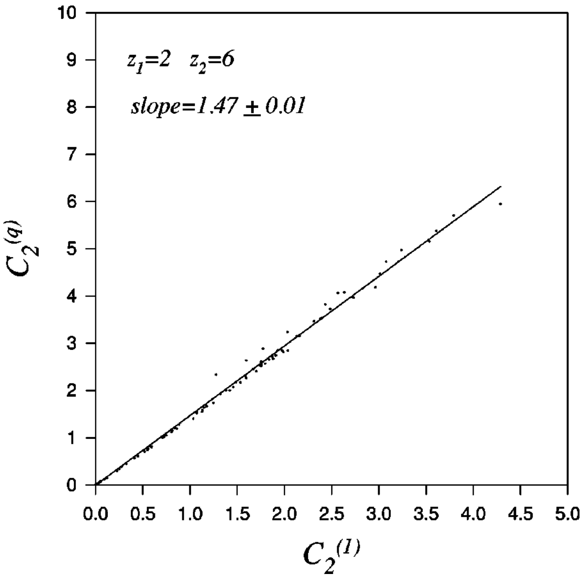

FIG. 1. The scaling between the standard second cumulant $C_2^{(1)}$ and the generalized second cumulant $C_2^{(q)}$ for a representative value of (z_1,z_2) pairs.

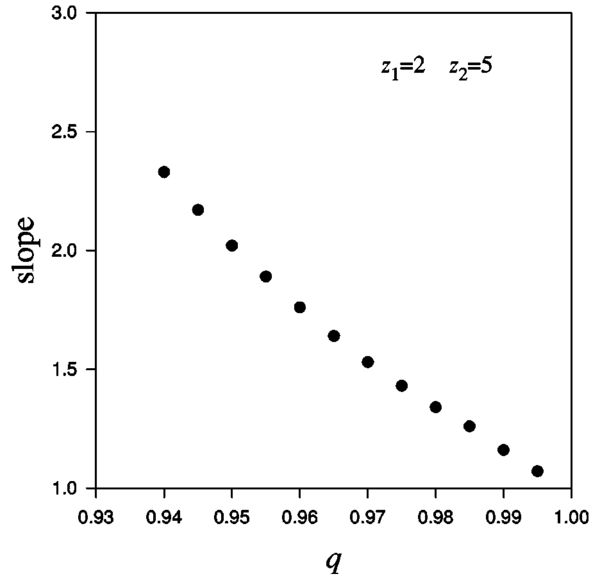

FIG. 2. The behavior of the slope of $C_2^{(1)}$ vs $C_2^{(q)}$ as a function of q index for a representative value of (z_1,z_2) pairs.

It has already been shown that for all the above-mentioned one-dimensional dissipative maps the values of q^* calculated within these two different methods are the same (within a good precision) for any given value of the inflection parameter z of the map used (for example, for the logisticlike maps $x_{i+1}=1-ax_i^z$, $z>1$). In addition to this, other important information that comes from all these reports is the determination of the behavior of q^* as a function of the inflection parameter z. As can be seen from Fig. 2 of [2] and from the table of [4], it seems that, when $z\to\infty$, q^* approaches unity. (In fact, it has been shown [15] that asymptotically the q^* values saturate before unity.)

The purpose of this paper is to show numerically whether such behavior is satisfied for asymmetric unimodal maps (AUMs)

$$x_{t+1}=\begin{cases}1-a|x_t|^{z_1} & \text{if } x_t\geqslant 0\\ 1-a|x_t|^{z_2} & \text{if } x_t\leqslant 0,\end{cases} \tag{5}$$

where $z_{1,2}>1$, $0<a\leqslant 2$, $-1\leqslant x_t\leqslant 1$, and $t=0,1,2,\ldots$. Unimodal maps have been studying intensively for the last two decades and some of the important work can be enumerated as follows: the investigation of nonuniversal behavior [16,17], scaling and multifractality properties [18], Feigenbaum theory [19,20], continuous invariant probability measures [21], complexity properties [22], and analysis of Feigenbaum attractors [23], among others [24,25]. Although all this and related work has provided a very good understanding of the dynamics of unimodal maps, up to now there has been no attempt to relate the class of maps defined by Eq. (5) to the nonextensivity parameter q of Tsallis statistics, as has already been done for other map families [1–4]. This will be the main goal of the present effort.

In fact, for the maps given in Eq. (5), unfortunately, the use of the two methods described above seems harder than usual [26]. Hence, we were not able to find satisfactory results for the prediction of q^* values of any inflection parameter pair (z_1,z_2). The problem of the first method might originate from prediction of the critical a_c values at the chaos threshold with enough precision, whereas the problem of the second method might be related to the numerical procedure used to estimate the $f(\alpha)$ curve, whose most rarefied region (α_{max}) is usually poorly sampled. Another possible fact that might cause our attempts to fail is the nonuniversal behavior of this subclass of AUMs as reported in [16,17] (for example, it is shown that these maps fail to exhibit the metric universality of Feigenbaum). Since values of q^* are not available for this subclass of AUMs, clearly it is not possible to see the behavior of q^* as a function of the (z_1,z_2) pairs by using the above-mentioned two methods. In this paper, in order to see this behavior, without finding the precise values of q^* for (z_1,z_2) pairs, we use another technique based on the very recent generalization of bit cumulants for chaotic

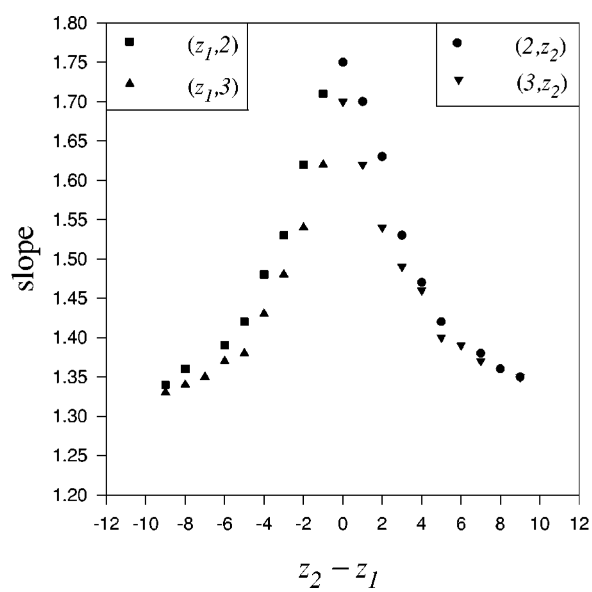

FIG. 3. The behavior of the slope of $C_2^{(1)}$ vs $C_2^{(q)}$ as a function of z_2-z_1 values for a family of four different (z_1,z_2) pairs.

systems within Tsallis statistics [27,28]. To explain this technique, let us recall the main results of [27,28]. The generalized second cumulant (or heat capacity) is given by

$$C_2^{(q)}=\frac{q}{(q-1)^2}[\langle\rho^{2q-1}\rangle-\langle\rho^q\rangle^2], \tag{6}$$

where ρ is the natural invariant density [29]. As $q\to 1$, we have

$$C_2^{(1)}=\langle(\ln\rho)^2\rangle-\langle\ln\rho\rangle^2, \tag{7}$$

which is equivalent to the standard definition of the second cumulant [29,30]. It has been shown [27,28] that there is a kind of scaling between $C_2^{(1)}$ and $C_2^{(q)}$ that is evident from a $C_2^{(1)}$ vs $C_2^{(q)}$ plot, where it is easily seen that most of the points can be fitted to a straight line. The slope of this line gives the scaling factor between $C_2^{(1)}$ and $C_2^{(q)}$ and also provides us a tool to fulfill our aim in this paper. In Fig. 1, we illustrate this slope plotting $C_2^{(1)}$ vs $C_2^{(q)}$ curve for a representative (z_1,z_2) pair. It is evident that most of the data points fall onto a straight line which yields a well-defined slope [in fact, we found that a few points deviate from this straight line especially when the value of the inflection parameter starts to be very different from the standard case (namely, $z_1=z_2$), but since such data points are very few, we prefer to calculate the slope using a linear regression—with an error of ± 0.01 for each estimation of the slope—to data points which fall onto this straight line]. As is evident from Fig. 2, when $q\to 1$, naturally $C_2^{(q)}\to C_2^{(1)}$ and thus the slope of the $C_2^{(1)}$ vs $C_2^{(q)}$ plot also tends to unity for any inflection pair. Although in q-generalized bit cumulant theory q is a free parameter, the above-mentioned slope constitutes another parameter that behaves exactly the same as q^* since it is evident that when $q\to 1$ naturally q^* values also tend linearly to unity. Therefore, now we can check the behavior of this slope as a function of (z_1,z_2) pairs and this allows us to estimate the dependence of the q^* values to (z_1,z_2) pairs *without knowing the exact values of* q^* for these pairs. Figure 3 represents this behavior for the $(2,z_2),(z_1,2),(3,z_2)$, and $(z_1,3)$ cases. It seems from the figure that as $z_2-z_1\to\pm\infty$ the above-mentioned slope (and thus the q^* index) approaches unity. This tendency is exactly the same as that observed for a family of logisticlike maps [2] and a family of circularlike maps [4], namely, as $z\to\infty$, q^* approaches unity.

Summing up, for the first time (to the best of our knowledge), we managed to study the asymmetric unimodal map defined in Eq. (5) in such a way that it became possible to relate it with the nonextensivity parameter q of Tsallis statistics. By studying the behavior of another parameter (which behaves the same as the q^* values) coming from the q-generalized bit cumulants, we were able to show that the dependence of the q^* index on the inflection parameter pairs (z_1,z_2) of these maps appears to be very similar to that observed for other one-dimensional maps reported so far. Although a clear understanding of this point will be available only after determination of the exact values of q^* for (z_1,z_2) pairs, which is still lacking, we hope that the present work will be considered as a first attempt in this line and will accelarate other studies, since it seems from the results of the present effort that this subclass of AUMs have different q^* values for different (z_1,z_2) pairs. In order to verify this, as already addressed in the literature for other maps [9,13,14], the study of the entropy increase at the edge of chaos for this kind of AUM would be a good candidate, and is no doubt worth investigating.

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